On the Group Classification of Systems of Two Linear Second-Order Ordinary Differential Equations with Constant Coefficients

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Abstract

The completeness of the group classification of systems of two linear second-order ordinary differential equations with constant coefficients is delineated in the paper. The new cases extend what has been done in the literature. These cases correspond to the type of equations where the commutative property of the coefficient matrices with respect to the dependent variables and the first-order derivatives in the considered system does not hold. A discussion of the results as well as a note on the extension to linear systems of second-order ordinary differential equations with more than two equations are given.

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1 Introduction

In this paper we consider the complete group classification of systems of two linear second-order ordinary differential equations with constant coefficients. Systems of second-order ordinary differential equations arise in the modeling of physical phenomena in the areas of physics, chemistry and mathematics. Of interest to the study of such systems is their symmetry properties. The founder of symmetry analysis of scalar ordinary differential equations, Sophus Lie [1], gave a complete group classification of a scalar ordinary differential equation of the form

$$y'' = f(x, y).$$

L.V.Ovsiannikov [2] later performed Lie's classification in a different way. This classification was obtained by directly solving the determining equations and using the equivalence transformations. The same approach was applied in [3] for the group classification of more general types of equations of the form $y'' = P_3(x, y; y')$, where $P_3(x, y; y')$ is a polynomial of third degree with respect to the first-order derivative y'. In the general case of a scalar ordinary differential equation $y'' = f(x, y, y')$ the application of the method that involves directly solving the determining equations gives rise to some difficulties. The group classification of such equations [4] is based on the enumeration of all possible Lie algebras of operators acting on the plane (x, y). Lie [5] gave the classification of all dissimilar Lie algebras (under complex change of variables) in two complex variables. Later on, in 1992, Gonzalez-Lopez et al. ordered the Lie classification of realizations of complex Lie algebras and extended it to the real case [6]. A large amount of results on the dimension and structure of symmetry algebras of linearizable ordinary differential equations is well-known (see [4,7,8,9,10,11,12]).

There are several papers where the group classification of systems of second-order ordinary differential equations was considered. The literature has dealt extensively with symmetry properties of a scalar ordinary differential equation but the group classification of a system of even two linear second-order equations with constant coefficients is not complete. In recent works [12,13,14,15,16] the authors focused on the study of systems of second-order ordinary differential equations with constant coefficients of the form

$$\mathbf{y}'' = M\mathbf{y}, \tag{1}$$

where M is a matrix with constant entries. In the present paper it is shown that these types of systems do not exhaust a set of all systems of linear second-order ordinary differential equations with constant coefficients.

In the general case of systems of two linear second-order ordinary differential equations, the more advanced results are obtained in [17], where using the

canonical form,

$$\mathbf{y}'' = \begin{pmatrix} a(x) & b(x) \\ c(x) & -a(x) \end{pmatrix} \mathbf{y},$$

presented several representatives of nonequivalent classes. However the list of all distinguished representatives of systems of two linear second-order ordinary differential equations was not obtained in [17]. Recently the complete group classification of linear systems of two second-order ordinary differential equations has been studied in [18].

Here the complete classification of linear systems of two second-order ordinary differential equations with constant coefficients is presented.

As far as we are aware the results found here are new and have not been reported in the literature.

The paper is organized as follows: In the first part we present the simplification of a linear system of second-order equations with constant coefficients. This is then followed by a preliminary study of linear systems of second-order ordinary differential equations with constant coefficients. The later part is followed by a complete treatment of linear systems of second-order ordinary differential equations with constant coefficients as well as the discussion of the results and conclusion in the final part of the paper.

2 Simplification of a system of linear equations

Let a linear system of second-order equations with constant coefficients be given by:

$$\mathbf{y}'' = A\mathbf{y}' + B\mathbf{y} + \mathbf{f}, \tag{2}$$

where A and B are constant matrices. Using a particular solution $\mathbf{y}_p$, one can reduce (2) to the homogeneous system

$$\mathbf{y}'' = A\mathbf{y}' + B\mathbf{y}. \tag{3}$$

2.1 Linear change of the dependent variables

Applying the change,

$$\mathbf{y} = C\mathbf{x},$$

where $C = C(t)$ is a nonsingular matrix, system (3) becomes

$$\mathbf{x}'' = \bar{A}\mathbf{x}' + \bar{B}\mathbf{x}, \tag{4}$$

where

$$\bar{A} = C^{-1}(AC - 2C'), \quad \bar{B} = C^{-1}(BC + AC' - C'').$$

If one chooses the matrix $C(t)$ such that

$$C' = \frac{1}{2}AC,$$

then

$$\bar{B} = C^{-1}\left(BC + \frac{1}{2}A^2C - \frac{1}{4}A^2C\right) = C^{-1}\left(B + \frac{1}{4}A^2\right)C.$$

The existence of the nonsingular matrix $C(t)$ is guaranteed by the existence of the solution of the Cauchy problem

$$C' = \frac{1}{2}AC, \quad C(0) = E,$$

where E is the unit matrix. This solution defines the matrix

$$C(t) = e^{\frac{t}{2}A},$$

which commutes with the matrix A:

$$AC = CA.$$

Let us study the case when the matrix $\bar{B}$ is constant. First of all notice that

$$\frac{d}{dt}\left(C^{-1}\right) = -C^{-1}C'C^{-1}.$$

Then one has

$$\begin{gathered}\tfrac{d}{dt}\bar{B} = \tfrac{d}{dt}\left(C^{-1}\left(B + \tfrac{1}{4}A^2\right)C\right)\\ = -C^{-1}ACC^{-1}\left(B + \tfrac{1}{4}A^2\right)C + C^{-1}\left(B + \tfrac{1}{4}A^2\right)AC\\ = -C^{-1}A\left(B + \tfrac{1}{4}A^2\right)C + C^{-1}\left(B + \tfrac{1}{4}A^2\right)AC = C^{-1}(BA - AB)C.\end{gathered}$$

Hence, the matrix $\bar{B}$ is constant if and only if the matrices A and B commute

$$AB = BA. \tag{5}$$

The papers cited in the literature [12,13,14,15,16], where the group classification of systems of linear equations with constant coefficients was considered, studied systems with constant matrix $\bar{B}$. Thus, even for systems of linear equations with constant coefficients there is no complete study of the group classification. The present paper fills this niche.

2.2 Example

Let us choose the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

then

$$AB - BA = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Notice that

$$A^2 = E.$$

Then

$$\begin{aligned} e^{sA} &= E + sA + \tfrac{s^2}{2!}A^2 + \tfrac{s^3}{3!}A^3 + \tfrac{s^4}{4!}A^4 + ... \\ &= \left(\tfrac{s^2}{2!} + \tfrac{s^4}{4!} + ... + \tfrac{s^{2n+1}}{(2n+1)!}...\right) E \\ &+ \left(sA + \tfrac{s^3}{3!}A^3 + ... + \tfrac{s^{2n+1}}{(2n+1)!}A^{2n+1} + ...\right) \\ &= \tfrac{1}{2}\left(\left(e^s + e^{-s}\right)E + \left(e^s - e^{-s}\right)\right)A. \end{aligned}$$

Hence,

$$C(t) = \varphi(t)E + \psi(t)A,$$

where $\varphi(2t) = \frac{1}{2}\left(e^t + e^{-t}\right)$ and $\psi(2t) = \frac{1}{2}\left(e^t - e^{-t}\right)$. One can check directly that

$$\bar{A} = 0, \quad \bar{B} = \frac{1}{4}\begin{pmatrix} (1+\varphi(2t)) & (2+\psi(2t)) \\ (2-\psi(2t)) & (1-\varphi(2t)) \end{pmatrix}.$$

It is obvious that the matrix $\bar{B}$ is not constant.

3 Preliminary study of linear systems of second-order ordinary differential equations with constant coefficients

Let us consider a linear system of second-order ordinary differential equations,

$$\mathbf{y}'' = A\mathbf{y}' + B\mathbf{y}, \tag{6}$$

where A and B are matrices with constant entries. For commuting matrices $AB = BA$ this system is reduced to the system of the form

$$\mathbf{y}'' = M\mathbf{y}, \tag{7}$$

which is completely studied in [12,13,14,15,16]. The case of noncommutative matrices is considered here.

3.1 Equivalence transformations

The general solution of the determining equations of the equivalence Lie group defines the equivalence Lie group, which corresponds to the generators

$$X_1^e = \partial_x, \ \ X_2^e = y\partial_y + z\partial_z,$$

$$\begin{aligned}
X_3^e &= z\partial_z - a_{12}\partial_{a_{12}} + a_{21}\partial_{a_{21}} - b_{12}\partial_{b_{12}} + b_{21}\partial_{b_{21}}, \\
X_4^e &= y\partial_z - a_{12}\partial_{a_{11}} - (a_{22} - a_{11})\partial_{a_{21}} + a_{12}\partial_{a_{22}} \\
&\quad -b_{12}\partial_{b_{11}} - (b_{22} - b_{11})\partial_{b_{21}} + b_{12}\partial_{b_{22}}, \\
X_5^e &= z\partial_y + a_{21}\partial_{a_{11}} + (a_{22} - a_{11})\partial_{a_{12}} - a_{21}\partial_{a_{22}} \\
&\quad +b_{21}\partial_{b_{11}} + (b_{22} - b_{11})\partial_{b_{12}} - b_{21}\partial_{b_{22}}, \\
X_6^e &= x\partial_x - a_{11}\partial_{a_{11}} - a_{12}\partial_{a_{12}} - a_{21}\partial_{a_{21}} - a_{22}\partial_{a_{22}} \\
&\quad -2b_{11}\partial_{b_{11}} - 2b_{12}\partial_{b_{12}} - 2b_{21}\partial_{b_{21}} - 2b_{22}\partial_{b_{22}}, \\
X_7^e &= x(y\partial_y + z\partial_z) + 2\partial_{a_{11}} + 2\partial_{a_{22}} \\
&\quad -a_{11}\partial_{b_{11}} - a_{12}\partial_{b_{12}} - a_{21}\partial_{b_{21}} - a_{22}\partial_{b_{22}}.
\end{aligned}$$

The transformations related with the generators X_2^e, X_3^e, X_4^e and X_5^e correspond to the linear change of the dependent variables $\bar{\mathbf{y}} = P\mathbf{y}$ with a constant nonsingular matrix P. The generators X_1^e and X_6^e define shifting and scaling x. The transformations corresponding to the generator X_7^e define the change

$$\begin{gathered}
\bar{y} = ye^{\tau x}, \ \ \bar{z} = ze^{\tau x}, \ \ \bar{a}_{11} = a_{11} + 2\tau, \ \ \bar{a}_{12} = a_{12}, \\
\bar{a}_{21} = a_{21}, \ \ \bar{a}_{22} = a_{22} + 2\tau, \ \ \bar{b}_{11} = b_{11} - a_{11}\tau - \tau^2, \\
\bar{b}_{12} = b_{11} - a_{12}\tau, \ \ \bar{b}_{21} = b_{21} - a_{21}\tau, \ \ \bar{b}_{22} = b_{22} - a_{22}\tau - \tau^2,
\end{gathered}$$

where τ is a group parameter. In matrix form the last set of transformations is rewritten as

$$\bar{\mathbf{y}} = e^{\tau x}\mathbf{y}, \quad \bar{A} = A + 2\tau I, \ \ \bar{B} = B - \tau A - \tau^2 I, \tag{8}$$

where I is the identical matrix.

3.2 Simplification of the matrix A

Let us apply the change $\bar{\mathbf{y}} = P\mathbf{y}$, where P is a nonsingular matrix with constant entries

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}.$$

Equations (6) become

$$\bar{\mathbf{y}}'' = \bar{A}\bar{\mathbf{y}}' + \bar{B}\bar{\mathbf{y}},$$

where

$$\bar{A} = PAP^{-1}, \quad \bar{B} = PBP^{-1}.$$

This means that the change $\bar{\mathbf{y}} = P\mathbf{y}$ reduces equation (6) to the same form with the matrices A and B changed. Using this change, the matrix A can be presented in the Jordan form. For a real-valued 2×2 matrix A, and if the matrix P also has real-valued entries, then the Jordan matrix is one of the following three types,

$$J_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad J_2 = \begin{pmatrix} a & c \\ -c & a \end{pmatrix}, \quad J_3 = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}, \tag{9}$$

where a, b and $c > 0$ are real numbers. By virtue of transformations (8) one can assume that $a = 0$. Notice also that using the dilation of x, one can reduce c to 1.

In these cases one obtains the following three cases:

$$\begin{aligned}
A = J_{1|a=0} \quad &: AB - BA = \begin{pmatrix} 0 & -b_{12}b \\ b_{21}b & 0 \end{pmatrix}; \\
A = J_{2|a=0,c=1} &: AB - BA = \begin{pmatrix} (b_{12} + b_{21}) & (b_{22} - b_{11}) \\ (b_{22} - b_{11}) & -(b_{12} + b_{21}) \end{pmatrix}; \\
A = J_{3|a=0} \quad &: AB - BA = \begin{pmatrix} b_{21} & (b_{22} - b_{11}) \\ 0 & -b_{21} \end{pmatrix};
\end{aligned}$$

where b_{ij} are entries of the matrix B.

This then gives the following for noncommutative matrices:

(a) in the case $A = J_1$ one can assume that $b_{12}b \neq 0$;

(b) in the case $A = J_2$ one has to assume that $(b_{12} + b_{21})^2 + (b_{22} - b_{11})^2 \neq 0$;

(c) in the case $A = J_3$ one has to assume that ${b_{21}}^2 + (b_{22} - b_{11})^2 \neq 0$.

3.3 Case $A = J_1$

In this case the matrix A is given as follows:

$$A = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}.$$

Let us denote $b = 4\lambda$, where $\lambda \neq 0$. The admitted generator has the form

$$X = C_1 \bar{X}_1 + C_2 X_2,$$

and the determining equations are reduced to the study of the equations

$$\begin{gathered} C_1(b_{11}^3 - 2b_{11}^2 b_{22} + 7b_{11}^2\lambda^2 + 2b_{11}b_{12}b_{21} + b_{11}b_{22}^2 - 6b_{11}b_{22}\lambda^2 \\ -56b_{11}\lambda^4 - 2b_{12}b_{21}b_{22} - b_{22}^2\lambda^2 + 8b_{22}\lambda^4 + 48\lambda^6) = 0, \end{gathered} \tag{10}$$

$$h_1 h_2 C_1 = 0, \tag{11}$$
$$(24\lambda^4 + 14b_{22}\lambda^2 - 6b_{11}\lambda^2 - 2b_{11}b_{22} + b_{12}b_{21} + 2b_{22}^2)C_1 = 0, \tag{12}$$
$$C_2 b_{21} = 0, \tag{13}$$
$$(4b_{22} + 15\lambda^2)C_2 = 0, \ \ (4b_{11} - \lambda^2)C_2 = 0, \tag{14}$$

where

$$\begin{gathered} h_1 = b_{11} + b_{22} + 2\lambda^2, \ \ h_2 = b_{22} - b_{11} + 4\lambda^2, \\ \bar{X}_1 = e^{-2\lambda x}\left(h_2(\partial_x - \lambda(y\partial_y - z\partial_z)) - 2\lambda b_{12} z\partial_y\right), \\ X_2 = e^{-\lambda x}\left(2\partial_x - \lambda(y\partial_y - 3z\partial_z)\right). \end{gathered}$$

Let us consider equations which admit the generator X_2. In this case $C_2 \neq 0$ and from equations (13)-(14) one finds that

$$b_{11} = \lambda^2/4, \quad b_{22} = -15\lambda^2/4, \quad b_{21} = 0, \tag{15}$$

or the matrix

$$B = \frac{1}{4}\begin{pmatrix} \lambda^2 & 4b_{12} \\ 0 & -15\lambda^2 \end{pmatrix}. \tag{16}$$

Notice that in this case $h_2 = 0$. Since in this case equations (10)-(12) are also satisfied, then there are two admitted generators

$$X_1 = e^{-2\lambda x} z\partial_y, \ \ X_2 = e^{-\lambda x}\left(2\partial_x - \lambda(y\partial_y - 3z\partial_z)\right). \tag{17}$$

If the matrix B is not of the form (16), then $C_2 = 0$, and for the existence of a nontrivial admitted generators one needs to require that $C_1 \neq 0$. Considering equation (11), first assume that $h_1 = 0$. Then equations (10) and (12) give

$$b_{11} = \lambda^2, \quad b_{22} = -3\lambda^2, \quad b_{21} = 0.$$

These conditions provide that $h_2 = 0$. Thus the case $h_2 = 0$ is the general case.

Assume now that $h_2 = 0$. The general solution of equations (10)-(12) is

$$b_{11} = b_{22} + 4\lambda^2, \quad b_{21} = 0,$$

or the matrix

$$B = \begin{pmatrix} b_{22} + 4\lambda^2 & b_{12} \\ 0 & b_{22} \end{pmatrix}.$$

In this case equations (14) are reduced to the equation

$$C_2(4b_{22} + 15\lambda^2) = 0.$$

Thus, if $4b_{22} + 15\lambda^2 = 0$, then one comes to the studied case (15), where the admitted generators are (17). If $4b_{22} + 15\lambda^2 \neq 0$, then there is the only admitted generator X_1.

In the summary the result is the following.

There are nontrivial admitted generators only if the matrix B has the form:

$$B = \begin{pmatrix} b_{22} + 4\lambda^2 & b_{12} \\ 0 & b_{22} \end{pmatrix}.$$

If $4b_{22} + 15\lambda^2 \neq 0$, then there is the only admitted generator

$$X_1 = e^{-2\lambda x} z\partial_y.$$

The extension of the Lie algebra, defined by the generator X_1, only occurs for $b_{22} = -15\lambda^2/4$. In this case the equations admit two generators

$$X_1 = e^{-2\lambda x} z\partial_y, \;\; X_2 = e^{-\lambda x}\left(2\partial_x - \lambda(y\partial_y - 3z\partial_z)\right).$$

3.4 Cases $A = J_2$ and $A = J_3$

In these cases

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

respectively. Calculations show that there are no extensions of the trivial generators in these cases.

4 Conclusion

This paper completes the study of the group classification of linear systems of two second-order ordinary differential equations with constant coefficients. In the literature the studied case reduced to $\mathbf{y}'' = M\mathbf{y}$ where M is a matrix with constant entries. This study can be extended to linear systems of second-order ordinary differential equations with more than two equations. For noncommutative matrices we found new cases where there are additional generators in addition to the generic ones, ∂_x and $y\partial_y + z\partial_z$. This implies that when considering the classification of such systems the first step should be to check whether the matrices A and B commute in (6). If they commute then the case is equivalent to the case (1) studied in the literature where M is a constant matrix whilst if A and B do not commute then the study in this paper applies and can be extended to deal with linear systems of second-order ordinary differential equations with more than two equations.

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